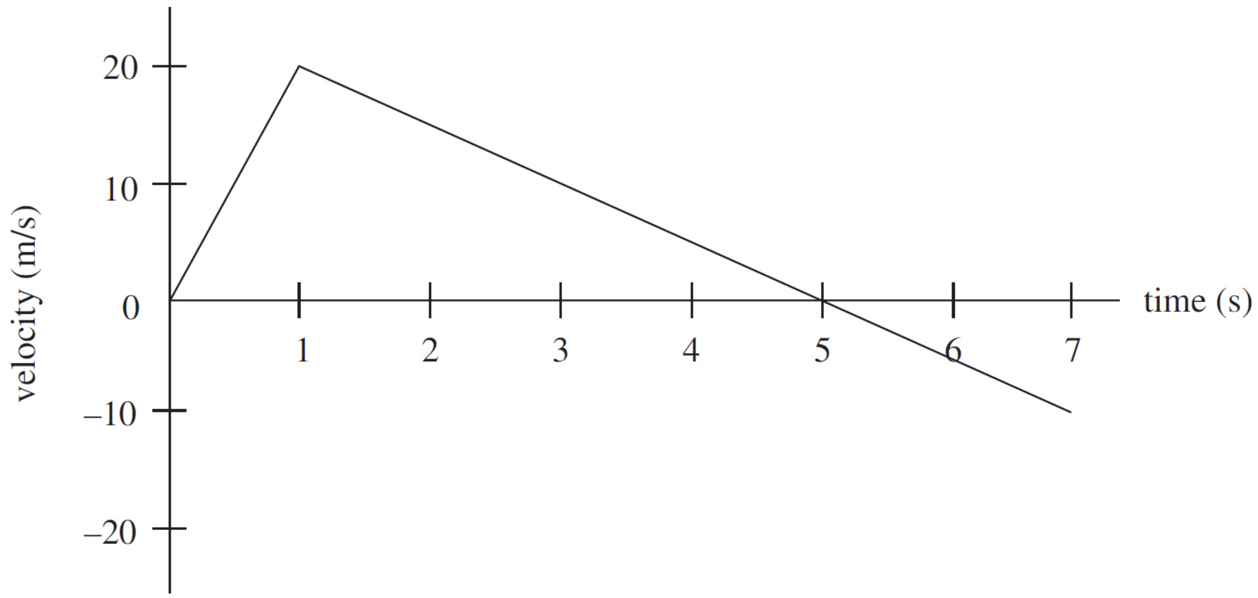
Section II: Free Response

1. This question concerns the motion of a car on a straight track; the car's velocity as a function of time is plotted below.



(a) Describe what happened to the car at time $t = 1$ s.

(b) How does the car's average velocity between time $t = 0$ and $t = 1$ s compare to its average velocity between times $t = 1$ s and $t = 5$ s?

(c) What is the displacement of the car from time $t = 0$ to time $t = 7$ s?

(d) Plot the car's acceleration during this interval as a function of time.

(e) Make a sketch of the object's position during this interval as a function of time. Assume that the car begins at $x = 0$.

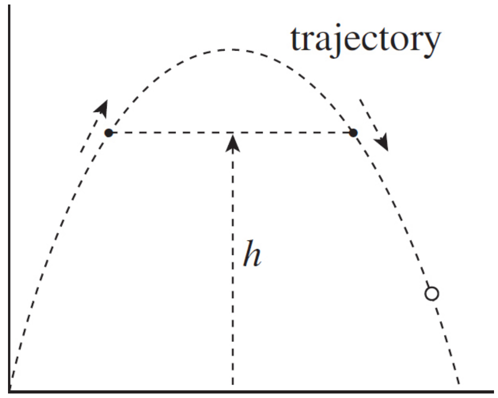
2. Consider a projectile moving in a parabolic trajectory under constant gravitational acceleration. Its initial velocity has magnitude v_0, and its launch angle (with the horizontal) is θ_0.

(a) Calculate the maximum height, H, of the projectile.

(b) Calculate the (horizontal) range, R, of the projectile.

(c) For what value of θ_0 will the range be maximized?

(d) If $0 < h < H$, compute the time that elapses between passing through the horizontal line of height h in both directions (ascending and descending); that is, compute the time required for the projectile to pass through the two points shown in this figure:



3. A cannonball is shot with an initial speed of 50 m/s at a launch angle of 40° toward a castle wall 220 m away. The height of the wall is 30 m. Assume that effects due to the air are negligible. (For this problem, use g = 9.8 m/s^2.)

(a) Show that the cannonball will strike the castle wall.

(b) How long will it take for the cannonball to strike the wall?

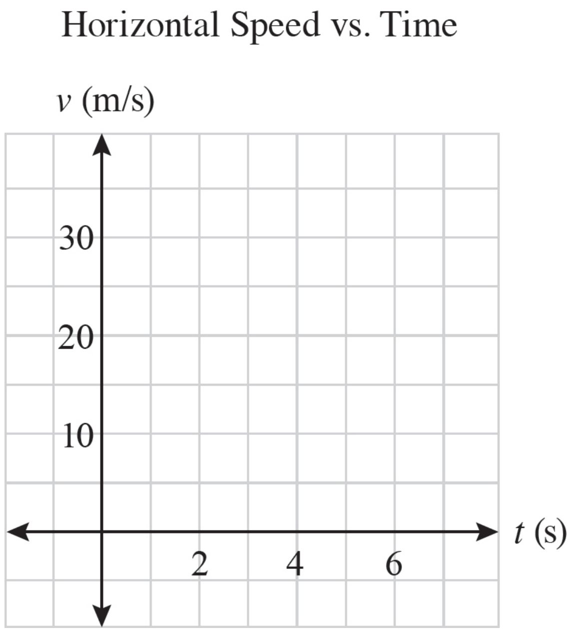
(c) At what height above the base of the wall will the cannonball strike?

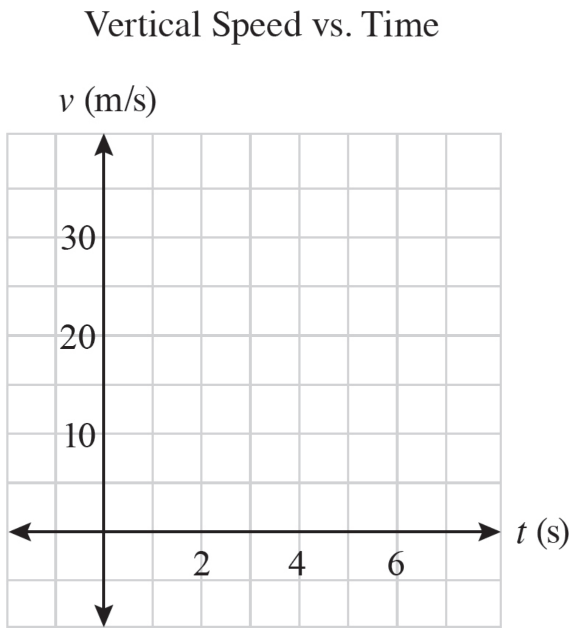
4. A cannonball is fired with an initial speed of 40 m/s and a launch angle of 30° from a cliff that is 25 m tall.

(a) What is the flight time of the cannonball?

(b) What is the range of the cannonball?

(c) In the two graphs below, plot the horizontal speed of the projectile versus time and the vertical speed of the projectile versus time (from the initial launch of the projectile to the instant it strikes the ground).





Section II: Free Response

1. (a)

At time $t = 1$ s, the car's velocity starts to decrease as the acceleration (which is the slope of the given velocity-versus-time graph) changes from positive to negative.

(b)

The average velocity between $t = 0$ and $t = 1$ s is $\frac{1}{2}(v_{t=0} + v_{t=1}) = \frac{1}{2}(0 + 20 \text{ m/s}) = 10 \text{ m/s}$, and the average velocity between $t = 1$ and $t = 5$ is $\frac{1}{2}(v_{t=1} + v_{t=5}) = \frac{1}{2}(20 \text{ m/s} + 0) = 10$ m/s. The two average velocities are the same.

(c)

The displacement is equal to the area bounded by the graph and the t-axis, taking areas above the t-axis as positive and those below as negative. In this case, the displacement from $t = 0$ to $t = 5$ s is equal to the area of the triangular region whose base is the segment along the t-axis from $t = 0$ to $t = 5$ s:

$$\Delta x\ (t = 0 \text{ to } t = 5 \text{ s}) = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(5 \text{ s})(20 \text{ m/s}) = 50 \text{ m}$$

The displacement from $t = 5$ s to $t = 7$ s is equal to the negative of the area of the triangular region whose base is the segment along the t-axis from $t = 5$ s to $t = 7$ s:

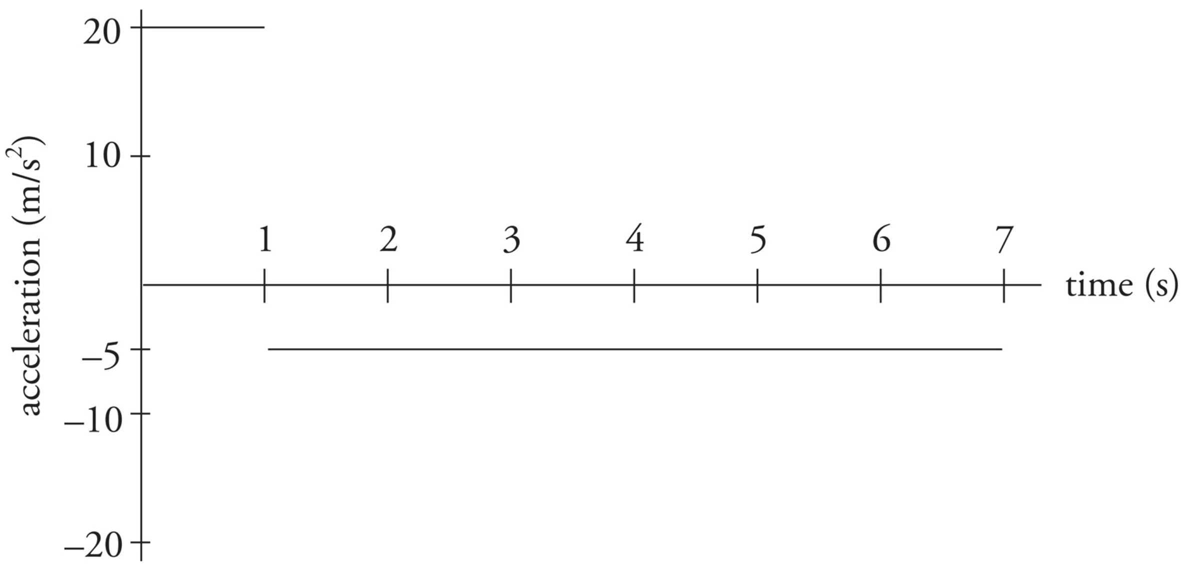
$$\Delta x\ (t = 5 \text{ s to } t = 7 \text{ s}) = -\frac{1}{2} \times \text{base} \times \text{height} = -\frac{1}{2}(2 \text{ s})(10 \text{ m/s}) = -10 \text{ m}$$

Therefore, the displacement from $t = 0$ to $t = 7$ s is

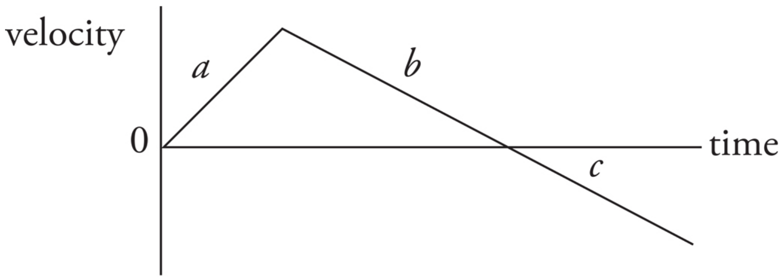
$$\Delta x\ (t = 0 \text{ to } t = 5 \text{ s}) + \Delta s\ (t = 5 \text{ s to } t = 7 \text{ s}) = 50 \text{ m} + (-10 \text{ m}) = 40 \text{ m}$$

(d)

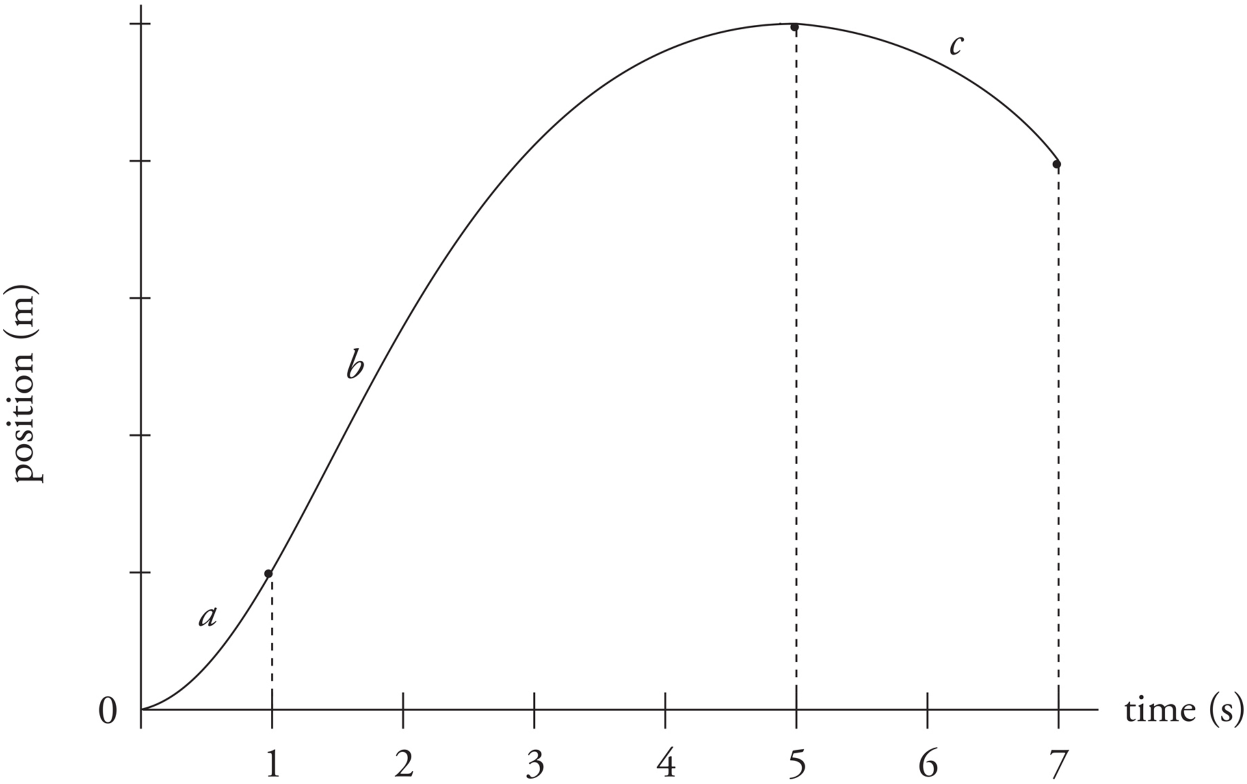
The acceleration is the slope of the velocity-versus-time graph. The segment of the graph from $t = 0$ to $t = 1$ s has a slope of $a = \Delta v/\Delta t = (20 \text{ m/s} - 0)/(1 \text{ s} - 0) = 20 \text{ m/s}^2$, and the segment of the graph from $t = 1$ s to $t = 7$ s has a slope of $a = \Delta v/\Delta t = (-10 \text{ m/s} - 20 \text{ s})/(7 \text{ s} - 1 \text{ s}) = -5 \text{ m/s}^2$. Therefore, the acceleration-versus-time graph is



(e)



Section *a* shows that the object is speeding up in the positive direction. Section *b* shows that the object is slowing down, yet still moving in the positive direction. At five seconds, the object has stopped for an instant. Section *c* shows that the object is moving in the negative direction and speeding up. The corresponding position-versus-time graph for each section would look like this:



2. (a)

The maximum height of the projectile occurs at the time at which its vertical velocity drops to zero:

$$v_y \stackrel{\text{set}}{=} 0 \quad \Rightarrow \quad v_{0y} - gt = 0 \quad \Rightarrow \quad t = \frac{v_{0y}}{g}$$

The vertical displacement of the projectile at this time is computed as follows:

$$\Delta y = v_{0y}t - \frac{1}{2}gt^2 \quad \Rightarrow \quad H = v_{0y}\frac{v_{0y}}{g} - \frac{1}{2}g\left(\frac{v_{0y}}{g}\right)^2 = \frac{v_{0y}^2}{2g} = \frac{v_0^2 \sin^2 \theta_0}{2g}$$

(b)

The total flight time is equal to twice the time computed in part (a):

$$t_t = 2t = 2\frac{v_{0y}}{g}$$

The horizontal displacement at this time gives the projectile's range:

$$\Delta x = v_{0x}t \quad \Rightarrow \quad R = v_{0x}t_t = \frac{v_{0x} \cdot 2v_{0y}}{g} = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \text{ or } \frac{v_0^2 \sin 2\theta_0}{g}$$

(c)

For any given value of v_0, the range,

$$\Delta x = v_{0x}t \quad \Rightarrow \quad R = v_{0x}t_t = \frac{v_{0x} \cdot 2v_{0y}}{g} = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g} \text{ or } \frac{v_0^2 \sin 2\theta_0}{g}$$

will be maximized when $\sin 2\theta_0$ is maximized. This occurs when $2\theta_0 = 90°$, that is, when $\theta_0 = 45°$.

(d)

Set the general expression for the projectile's vertical displacement equal to h and solve for the two values of t (assuming that $g = +10\text{ m/s}^2$):

$$v_{0y}t - \frac{1}{2}gt^2 \overset{\text{set}}{=} h \quad \Rightarrow \quad \frac{1}{2}gt^2 - v_{0y}t + h = 0$$

Applying the quadratic formula, find that

$$t = \frac{v_{0y} \pm \sqrt{(-v_{0y})^2 - 4(\frac{1}{2}g)(h)}}{2(\frac{1}{2}g)} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 - 2gh}}{g}$$

Therefore, the two times at which the projectile crosses the horizontal line at height $h$are

$$t_1 = \frac{v_{0y} - \sqrt{v_{0y}^2 - 2gh}}{g} \quad \text{and} \quad t_2 = \frac{v_{0y} + \sqrt{v_{0y}^2 - 2gh}}{g}$$

so the amount of time that elapses between these events is

$$\Delta t = t_2 - t_1 = \frac{2\sqrt{v_{0y}^2 - 2gh}}{g}$$

3. (a)

The cannonball will certainly reach the wall (which is only 220 m away) since the ball's range is

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(50\text{ m/s})^2 \sin 2(40°)}{9.8\text{ m/s}^2} = 251\text{ m}$$

You simply need to make sure that the cannonball's height is less than 30 m at the point where its horizontal displacement is 220 m (so that the ball actually hits the wall rather than flying over it). To do this, find the time at which $x = 220$ m by first writing

$$x = v_{0x}t \quad \Rightarrow \quad t = \frac{x}{v_{0x}} = \frac{x}{v_0 \cos\theta_0} \qquad (1)$$

Thus, the cannonball's vertical position can be written in terms of its horizontal position as follows:

$$y = v_{0y}t - \tfrac{1}{2}gt^2 = v_0 \sin\theta_0 \frac{x}{v_0 \cos\theta_0} - \tfrac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2$$
$$= x\tan\theta_0 - \frac{gx^2}{2v_0^2 \cos^2\theta_0} \qquad (2)$$

Substituting the known values for x, θ_0, g, and v_0, you get

$$y(\text{at } x = 220 \text{ m}) = (220 \text{ m})\tan 40° - \frac{(9.8 \text{ m/s}^2)(220 \text{ m})^2}{2(50 \text{ m/s})^2 \cos^2 40°}$$
$$= 23 \text{ m}$$

This is indeed less than 30 m, as desired.

(b)

From Equation (1) derived in part (a),

$$t = \frac{x}{v_0 \cos\theta_0} = \frac{220 \text{ m}}{(50 \text{ m/s})\cos 40°} = 5.7 \text{ s}$$

(c)

The height at which the cannonball strikes the wall was determined in part (a) to be 23 m.

4. (a)

For parabolic trajectories, the total flight time can be determined by doubling the amount of time it takes for the projectile to reach its apex. However, this trajectory is NOT parabolic. As the initial position is 25 m, the final position is 0 m, the initial velocity is $v_0 = v_0 \sin\theta = 40 \sin 30° = 40\left(\frac{1}{2}\right) = 20$ m/s, and the acceleration is -10 m/s^2; the missing variable is the final velocity, so the flight time of the cannonball can be computed using Big Five #3:

$$x_y = x_0 + v_0t + \frac{1}{2}at^2$$

$$0 = 25 \text{ m} + 20 \text{ m/s} \cdot t + \frac{1}{2}(-10 \text{ m/s}^2) \cdot t^2$$

$$0 = 25 \text{ m} + 20 \text{ m/s} \cdot t - 5 \text{ m/s}^2 \cdot t^2$$

$$0 = -5 \text{ m/s}^2 \cdot t^2 + 20 \text{ m/s} \cdot t + 25 \text{ m}$$

Applying the quadratic formula, find that

$$t = \frac{-20 \pm \sqrt{20^2 - 4(25)(-5)}}{2(-5)} = \frac{-20 \pm \sqrt{400 + 500}}{-10} = \frac{-20 \pm \sqrt{900}}{-10} = \frac{-20 \pm 30}{-10}$$

$$t = -1 \text{ s or } t = 5 \text{ s}$$

As time cannot be a negative value, the flight time of the cannonball is 5 seconds.

(b)

As the horizontal speed of a projectile is constant, the range is given by

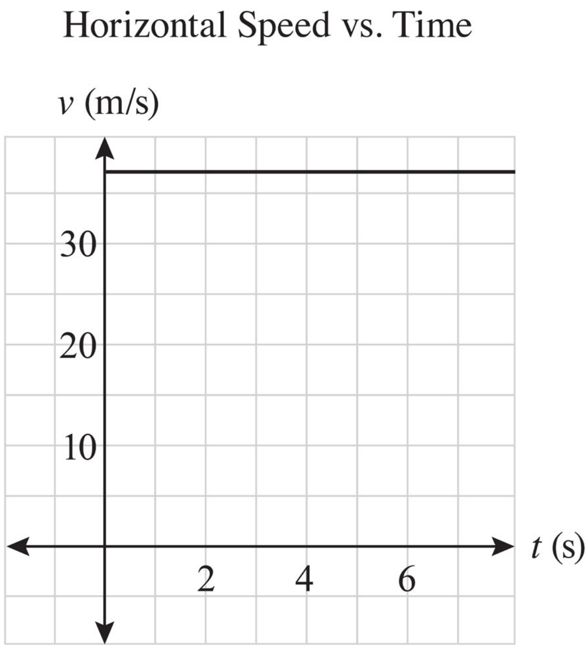
$$\Delta x = v_{ox}t = 40\text{ m/s} \cdot \cos 30° \cdot 5\text{ s} = 173\text{ m}$$

(c)

The cannonball is fired with an initial horizontal velocity of

$$v_{ox} = 40\text{ m/s} \cdot \cos 30° = 34.6\text{ m/s}$$

As there is no horizontal acceleration, the horizontal speed of the projectile remains constant throughout the entire flight, leading to a flat line with a y-value of 34.6 m/s over the flight time of 5 s. The horizontal speed-versus-time graph can then be plotted.

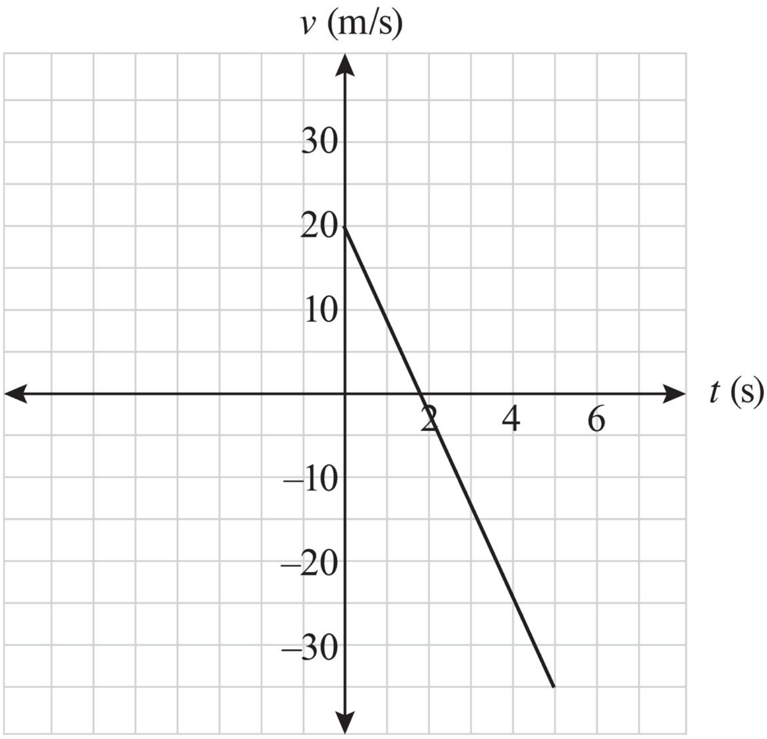


The initial vertical velocity of the cannonball is

$$v_{ox} = 40\text{ m/s} \cdot \sin 30° = 20\text{ m/s}$$

In the vertical direction, there is a constant acceleration due to gravity (a = 10 m/s²). As the slope of the velocity-versus-time graph is equal to the acceleration, the resulting velocity-versus-time graph is a linear line with a negative slope since acceleration due to gravity points downward. As the magnitude of the acceleration of gravity is 10 m/s², the vertical velocity of the cannonball thus decreases by 10 m/s every second. With an initial velocity of 20 m/s, this means that after 2 s, the vertical velocity of the projectile is 0 m/s. After a total of 5 s, the vertical velocity of the projectile is −30 m/s. This can be visualized in the vertical velocity-versus-time graph below:

Vertical Velocity vs. Time



As the problem asks for the graph of the vertical speed versus time, the absolute value of the graph must be drawn to get the correct plot.

Vertical Speed vs. Time