Oscillations

"There is nothing so small, as to escape our inquiry; hence there is a new visible World discovered to the understanding."

—Robert Hooke

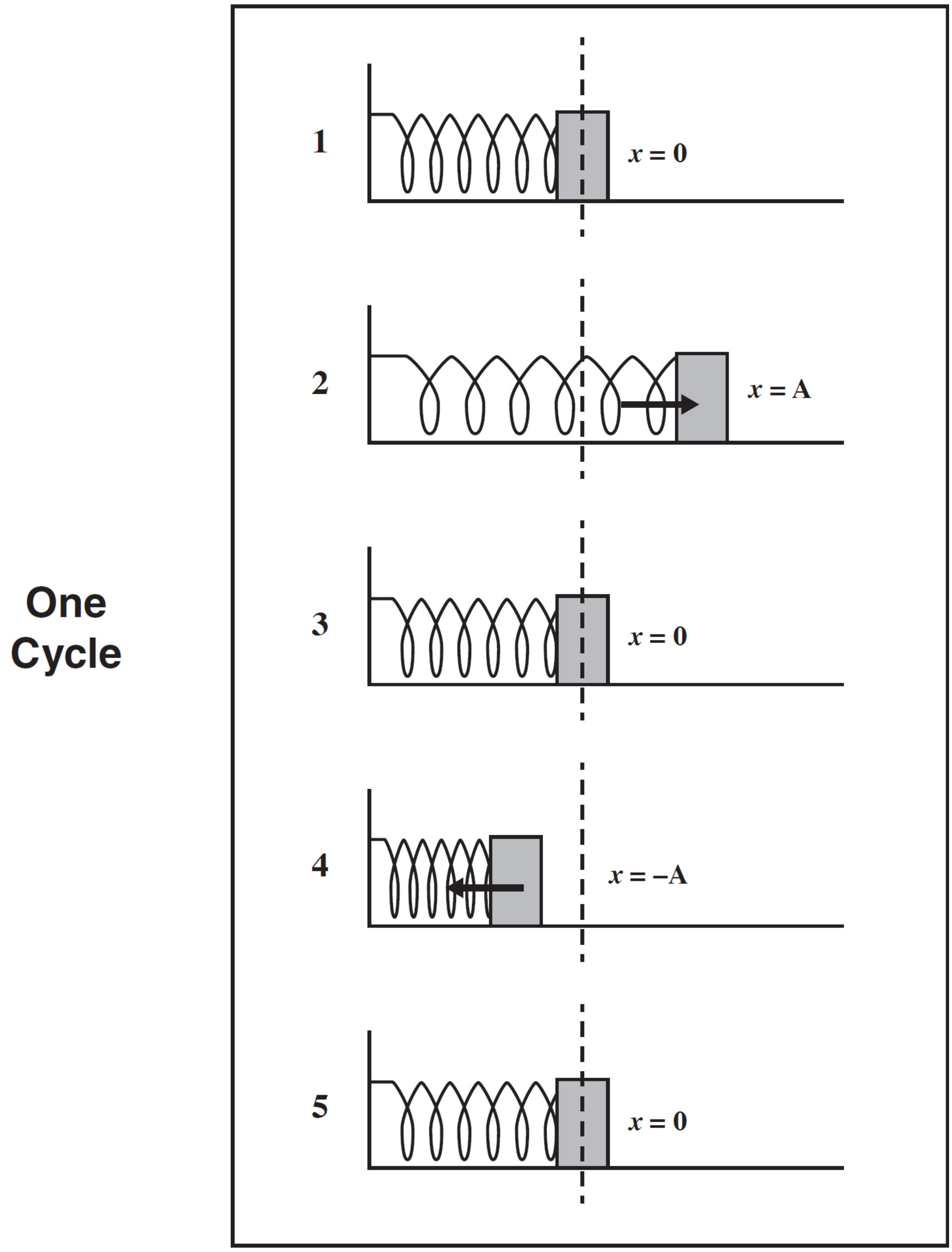
Seventeenth-century British physicist Robert Hooke helped pave the way for simple harmonic motion. Previously, Newton's laws of static equilibrium made it possible to show a relationship between stress and strain for complex objects. Building upon these, he developed laws on simple harmonic motion that are named after him today—Hooke's Law. Hooke is credited as the discoverer of the concise mathematical relationship of a spring.

In this section, we will focus on periodic motion that's straightforward which will help to describe oscillations and other simple harmonic motions.

SIMPLE HARMONIC MOTION

In the series of diagrams below is a fixed block on the left side of a wall. When the spring is neither stretched nor compressed (when it sits at its natural length), it is said to be in its equilibrium position. When the block is in equilibrium, the net force on the block is zero. We label this position as $x = 0$.

Let's start with a spring at rest (Diagram 1). First, we pull the block to the right, where it will experience a force pulling back toward equilibrium (Diagram 2). This force brings the block back through the equilibrium position (Diagram 3), and the block's momentum carries it past that point to location $x = -A$. At this point, the block will again be experiencing a force that pushes it toward the equilibrium position (Diagram 4). Once again, the block passes through the equilibrium position, but it's traveling to the right this time (Diagram 5). If this is taking place in ideal conditions (no friction), this back-and-forth motion will continue indefinitely and the block will oscillate from these positions in the same amount of time. The oscillations of the block at the end of this spring provide us with the physical example of simple harmonic motion (often abbreviated SHM).



The Dynamics of SHM

Force

Since the block is accelerating and decelerating, there must be some force that is making it do so. In the case of a spring, the spring exerts a force on the block that is proportional to its displacement from its equilibrium point. Setting the equilibrium point as $x = 0$, the force exerted by the spring is

$$F = -kx$$

This is known as Hooke's Law. The proportionality constant, k, is called the **spring constant**and tells us how strong the spring is. The greater the value of k, the stiffer and stronger the spring is. The minus sign in Hooke's Law tells us that the force is a restoring force. A restoring force simply means that the force wants to return the object back to its equilibrium position. Hence, in the previous diagrams, when the block was on the left (when the position is negative), the force exerted was to the right; and when the block was on the right (when the position is positive), the force exerted was to the left. In all cases of the extreme left or right, the spring has a tendency to return to its original length or equilibrium position. This force helps to maintain the oscillations.

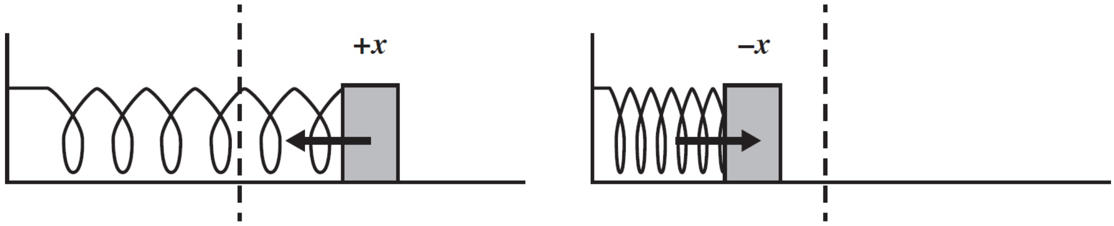
> **Example 1** A 12 cm-long spring has a force constant (k) of 400 N/m. How much force is required to stretch the spring to a length of 14 cm?

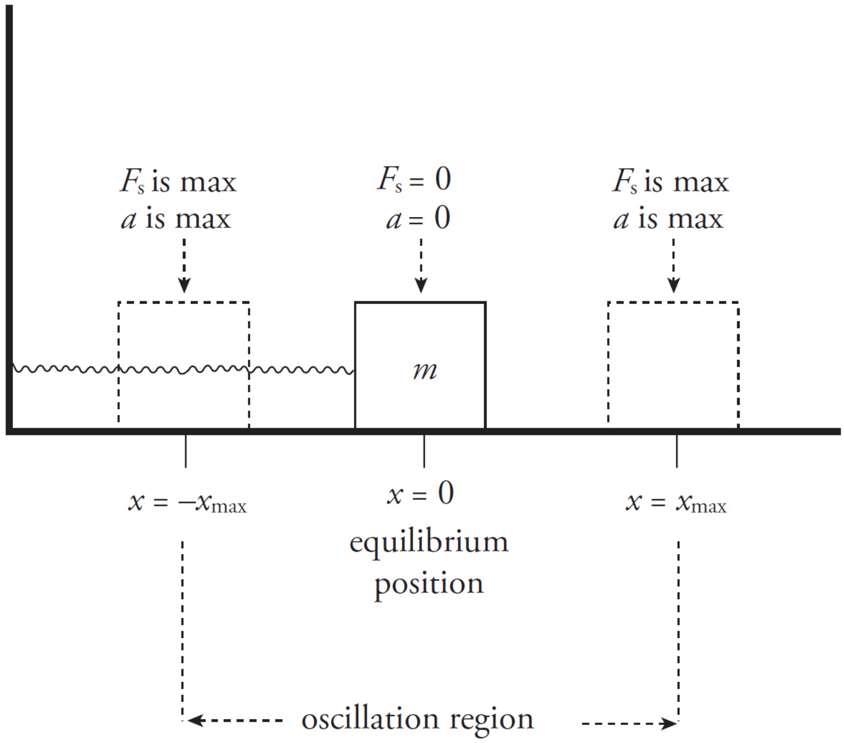
Solution. The displacement of the spring has a magnitude of 14 – 12 = 2 cm = 0.02 m so, according to Hooke's Law, the spring exerts a force of magnitude $F = kx = (400 \text{ N/m})(0.02 \text{ m}) = 8$ N. Therefore, we'd have to exert this much force to keep the spring in this stretched state.

During the oscillation, the force on the block is zero when the block is at equilibrium (the point we designate as $x = 0$). This is because Hooke's Law says that the strength of the spring's restoring force is given by the equation $F = kx$, so $F = 0$ at equilibrium. The acceleration of the block is also equal to zero at $x = 0$, since $F = 0$ at $x = 0$ and $a = F/m$. At the endpoints of the oscillating region, where the block's displacement, x, has the greatest magnitude, the restoring force and the magnitude of the acceleration are both at their maximums.

Amplitude

The maximum displacement from equilibrium is called the amplitude of oscillation and is denoted by A. So instead of writing $x = x_{max}$, we write $x = A$ ($x = -x_{max}$ is $x = -A$). This number tells us how far to the left and right of equilibrium the block will travel.





SHM in Terms of Energy

Another way to describe the block's motion is in terms of energy transfers. A stretched or compressed spring stores **elastic potential energy,** which is transformed into kinetic energy (and back again); this shuttling of energy between potential and kinetic causes the oscillations. For a spring with spring constant *k*, the elastic potential energy it possesses—relative to its equilibrium position—is given by the following equation:

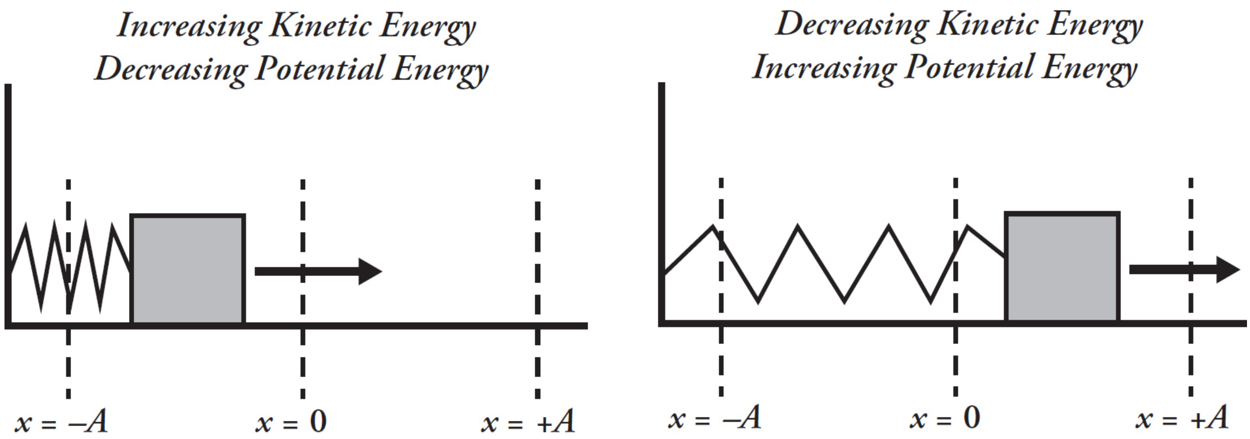
$$U_s = \frac{1}{2} kx^2$$

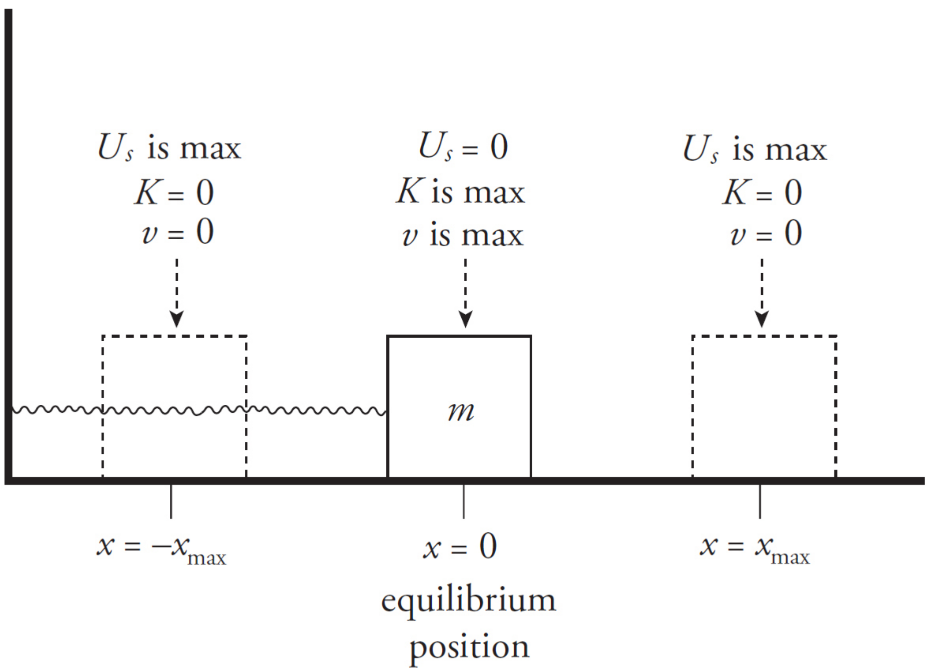
Notice that the farther you stretch or compress a spring, the more work you have to do, and, as a result, the more potential energy that's stored.

Maximum Potential Energy
The maximum potential energy is when $x = A$ *or* $x = -A$.

In terms of energy transfers, we can describe the block's oscillations as follows. When you initially pull the block out, you increase the elastic potential energy of the system. Upon releasing the block, this potential energy turns into kinetic energy, and the block moves. As it passes through equilibrium, $U_S = 0$, all the energy is kinetic. Then, as the block continues through equilibrium, it compresses the spring and the kinetic energy is transformed back into elastic potential energy.

By Conservation of Mechanical Energy, the sum $K + U_S$ is a constant. Therefore, when the block reaches the endpoints of the oscillation region (that is, when $x = \pm x_{max}$), U_S is maximized, so *K*must be minimized; in fact, $K = 0$ at the endpoints. As the block is passing through equilibrium, $x = 0$, so $U_S = 0$ and K is maximized.





> **Example 2** A block of mass $m = 0.05$ kg oscillates on a spring whose force constant k is 500 N/m. The amplitude of the oscillations is 4.0 cm. Calculate the maximum speed of the block.

Solution. First, let's get an expression for the maximum elastic potential energy of the system:

$$U_S = \frac{1}{2}kx^2 \quad \Rightarrow \quad U_{S,\,\max} = \frac{1}{2}kx_{\max}^2 = \frac{1}{2}kA^2$$

When all this energy has been transformed into kinetic energy—which, as we discussed earlier, occurs just as the block is passing through equilibrium—the block will have a maximum kinetic energy and maximum speed of

$$U_{S,\,max} \rightarrow K_{max} \quad \Rightarrow \quad \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$$

$$v_{max} = \sqrt{\frac{kA^2}{m}}$$

$$= \sqrt{\frac{(500\text{ N/m})(0.04\text{ m})^2}{0.05\text{ kg}}}$$

$$= 4\text{ m/s}$$

Maximum Velocity

Only the maximum velocity can be calculated in a spring with this method. The AP Physics 1 Exam will not ask you to calculate other velocities at other points because this is not uniform accelerated motion.

Example 3 A block of mass m = 2.0 kg is attached to an ideal spring of force constant k = 500 N/m. The amplitude of the resulting oscillations is 8.0 cm. Determine the total energy of the oscillator and the speed of the block when it's 4.0 cm from equilibrium.

Solution. The total energy of the oscillator is the sum of its kinetic and potential energies. By Conservation of Mechanical Energy, the sum $K + U_S$ is a constant, so if we can determine what this sum is at some point in the oscillation region, we'll know the sum at every point. When the block is at its amplitude position, x = 8 cm, its speed is zero; so at this position, E is easy to figure out:

$$E = K + U_S = 0 + \frac{1}{2}kA^2 = \frac{1}{2}(500\text{ N/m})(0.08\text{ m})^2 = 1.6\text{ J}$$

This gives the total energy of the oscillator at *every* position. At any position x, we have

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$$

$$v = \sqrt{\frac{E - \frac{1}{2}kx^2}{\frac{1}{2}m}}$$

so when we substitute in the numbers, we get

$$v = \sqrt{\frac{E - \frac{1}{2}kx^2}{\frac{1}{2}m}} = \sqrt{\frac{(1.6\text{ J}) - \frac{1}{2}(500\text{ N/m})(0.04\text{ m})^2}{\frac{1}{2}(2.0\text{ kg})}}$$

$$= 1.1\text{ m/s}$$

Example 4 A block of mass m = 8.0 kg is attached to an ideal spring of force constant k = 500 N/m. The block is at rest at its equilibrium position. An impulsive force acts on the block, giving it an initial speed of 2.0 m/s. Find the amplitude of the resulting oscillations.

Solution. The block will come to rest when all of its initial kinetic energy has been transformed into the spring's potential energy. At this point, the block is at its maximum displacement from equilibrium, at one of its amplitude positions, and

$$K_i + U_i = K_{ff} + U$$

$$\frac{1}{2}mv_i^2 + 0 = 0 + \frac{1}{2}kA^2$$

$$A = \sqrt{\frac{mv_i^2}{k}}$$

$$= \sqrt{\frac{(8.0 \text{ kg})(2.0 \text{ m/s})^2}{500 \text{ N/m}}}$$

$$= 0.25 \text{ m}$$

Period and Frequency

As you watch the block oscillate, you should notice that it repeats each **cycle** of oscillation in the same amount of time. A cycle is a *round-trip*: for example, from position $x = A$ over to $x = -A$ and back again to $x = A$. The amount of time it takes to complete a cycle is called the **period** of the oscillations, or T. If T is short, the block is oscillating rapidly, and if T is long, the block is oscillating slowly.

Another way of indicating the rapidity of the oscillations is to count the number of cycles that can be completed in a given time interval; the more completed cycles, the more rapid the oscillations. The number of cycles that can be completed per unit time is called the **frequency** of the oscillations, or f, and frequency is expressed in cycles per second. One cycle per second is one **hertz** (abbreviated **Hz).** Do not confuse lowercase f (frequency) with uppercase F (Force).

Two Peas in a Pod

If you have the period, you can always get the frequency and vice-versa. Note that the period and the frequency are reciprocals of each other.

One of the most basic equations of oscillatory motion expresses the fact that the period and frequency are reciprocals of each other:

$$\text{period} = \frac{\#\text{ seconds}}{\text{cycle}} \quad \text{while} \quad \text{frequency} = \frac{\#\text{ cycles}}{\text{second}}$$

Therefore,

$$T = \frac{1}{f} \quad \text{and} \quad f = \frac{1}{T}$$

Example 5 A block oscillating on the end of a spring moves from its position of maximum spring stretch to maximum spring compression in 0.25 s. Determine the period and frequency of this motion.

Solution. The period is defined as the time required for one full cycle. Moving from one end of the oscillation region to the other is only half a cycle. Therefore, if the block moves from its position of maximum spring stretch to maximum spring compression in 0.25 s, the time required for a full cycle is twice as much; $T = 0.5$ s. Because frequency is the reciprocal of period, the frequency of the oscillations is $f = 1/T = 1/(0.5 \text{ s}) = 2$ Hz.

Example 6 A student observing an oscillating block counts 45.5 cycles of oscillation in one minute. Determine its frequency (in hertz) and period (in seconds).

Solution. The frequency of the oscillations, in hertz (which is the number of cycles per second), is

$$f = \frac{45.5 \text{ cycles}}{\text{min}} \times \frac{1 \text{ min}}{60 \text{ s}} = \frac{0.758 \text{ cycles}}{\text{s}} = 0.758 \text{ Hz}$$

Therefore,

$$T = \frac{1}{f} = \frac{1}{0.758 \text{ Hz}} = 1.32 \text{ s}$$

One of the defining properties of the spring-block oscillator is that the frequency and period can be determined from the mass of the block and the force constant of the spring. The equations are as follows:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{and} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

A useful mnemonic for remembering this equation is to go in alphabetical order (clockwise) for frequency (*f* to *k* to *m*) and reverse alphabetical for period (*T* to *m* to *k*).

Let's analyze these equations. Suppose we had a small mass on a very stiff spring; then intuitively, we would expect that this strong spring would make the small mass oscillate rapidly, with high frequency and short period. Both of these predictions are substantiated by the equations above, because if m is small and k is large, then the ratio k/m is large (high frequency) and the ratio m/k is small (short period).

Example 7 A block of mass $m = 2.0$ kg is attached to a spring whose force constant, k, is 300 N/m. Calculate the frequency and period of the oscillations of this spring-block system.

Solution. According to the equations above,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{300 \text{ N/m}}{2.0 \text{ kg}}} = 1.9 \text{ Hz}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.0 \text{ kg}}{300 \text{ N/m}}} = 0.51 \text{ s}$$

Notice that $f \approx 2$ Hz and $T \approx 0.5$ s, and that these values satisfy the basic equation $T = 1/f$.

Example 8 A block is attached to a spring and set into oscillatory motion, and its frequency is measured. If this block were removed and replaced by a second block with 1/4 the mass of the first block, how would the frequency of the oscillations compare to that of the first block?

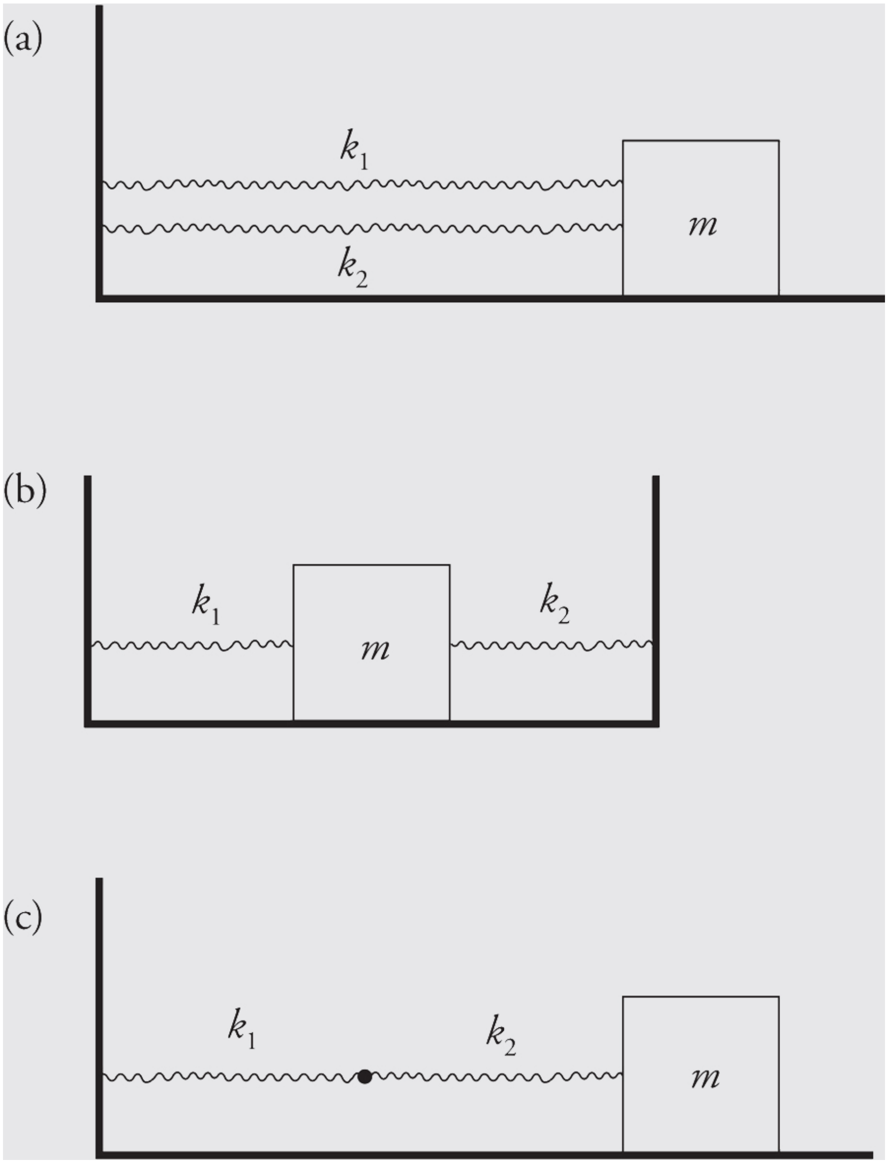
Solution. Since the same spring is used, k remains the same. According to the equation given on the previous page, f is inversely proportional to the square root of the mass of the block: $f \propto 1/\sqrt{m}$. Therefore, if m decreases by a factor of 4, then f increases by a factor of $\sqrt{4} = 2$.

The equations we saw on the previous page for the frequency and period of the spring-block oscillator do not contain A, the amplitude of the motion. In simple harmonic motion, *both the frequency and the period are independent of the amplitude*. The reason that the frequency and period of the spring-block oscillator are independent of amplitude is that F, the strength of the restoring force, is proportional to x, the displacement from equilibrium, as given by Hooke's Law: $F_S = -kx$.

Example 9 A student performs an experiment with a spring-block simple harmonic oscillator. In the first trial, the amplitude of the oscillations is 3.0 cm, while in the second trial (using the same spring and block), the amplitude of the oscillations is 6.0 cm. Compare the values of the period, frequency, and maximum speed of the block between these two trials.

Solution. If the system exhibits simple harmonic motion, then the period and frequency are independent of amplitude. This is because the same spring and block were used in the two trials, so the period and frequency will have the same values in the second trial as they had in the first. But the maximum speed of the block will be greater in the second trial than in the first. Since the amplitude is greater in the second trial, the system possesses more total energy ($E = \frac{1}{2}kA^2$). So when the block is passing through equilibrium (its position of greatest speed), the second system has more energy to convert to kinetic, meaning that the block will have a greater speed. In fact, from Example 2, we know that $v_{max} = A\sqrt{k/m}$ so, since A is twice as great in the second trial than in the first, v_{max} will be twice as great in the second trial than in the first.

Example 10 For each of the following arrangements of two springs, determine the **effective spring constant,** k_{eff}. This is the force constant of a single spring that would produce the same force on the block as the pair of springs shown in each case.



(d) Determine k_{eff} in each of these cases if $k_1 = k_2 = k$.

Solution.

(a) Imagine that the block was displaced a distance x to the right of its equilibrium position. Then the force exerted by the first spring would be $F_1 = -k_1x$ and the force exerted by the second spring would be $F_2 = -k_2x$. The net force exerted by the springs would be

$$F_1 + F_2 = -k_1x + (-k_2x) = -(k_1 + k_2)x$$

Since $F_{eff} = -(k_1 + k_2)x$, we see that $k_{eff} = k_1 + k_2$.

(b) Imagine that the block was displaced a distance x to the right of its equilibrium position. Then the force exerted by the first spring would be $F_1 = -k_1x$ and the force exerted by the second spring would be $F_2 = -k_2x$. The net force exerted by the springs would be

$$F_1 + F_2 = -k_1x + -k_2x = -(k_1 + k_2)x$$

As in part (a), we see that, since $F_{eff} = -(k_1 + k_2)x$, we get $k_{eff} = k_1 + k_2$.

(c) Imagine that the block was displaced a distance x to the right of its equilibrium position. Let x_1 be the distance that the first spring is stretched, and let x_2 be the distance that the second spring is stretched. Then $x = x_1 + x_2$. But $x_1 = -F/k_1$ and $x_2 = -F/k_2$, so

$$\frac{-F}{k_1} + \frac{-F}{k_2} = x$$

$$-F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = x$$

$$F = -\left(\frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}\right)x$$

$$F = -\frac{k_1k_2}{k_1 + k_2}x$$

Therefore,

$$k_{eff} = \frac{k_1k_2}{k_1 + k_2}$$

(d) If the two springs have the same force constant, that is, if $k_1 = k_2 = k$, then in the first two cases, the pairs of springs are equivalent to one spring that has twice their force constant: $k_{eff} = k_1 + k_2 = k + k = 2k$. In (c), the pair of springs is equivalent to a single spring with half their force constant:

$$k_{eff} = \frac{k_1k_2}{k_1 + k_2} = \frac{kk}{k + k} = \frac{k^2}{2k} = \frac{k}{2}$$

Spring-Block Summary

We can summarize the dynamics of oscillations in this table:

	$x = -A$	$x = 0$	$x = +A$
Magnitude of Restoring Force	MAX	0	MAX
Magnitude of Acceleration	MAX	0	MAX
Potential Energy (U) of Spring	MAX	0	MAX
Kinetic Energy (K) of Block	0	MAX	0
Speed (v) of Block	0	MAX	0

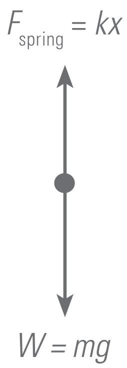
Fortunately, this same table applies to springs, pendulums, and waves. All simple harmonic motion follows this cycle.

THE SPRING-BLOCK OSCILLATOR: VERTICAL MOTION

So far we've looked at a block sliding back and forth on a horizontal table, but the block could also oscillate vertically. The only difference would be that gravity would cause the block to move downward, to an equilibrium position at which, in contrast with the horizontal SHM we've examined, the spring would not be at its natural length. Of course, in calculating energy, the gravitational potential energy (*mgh*) must be included.

Another Approach

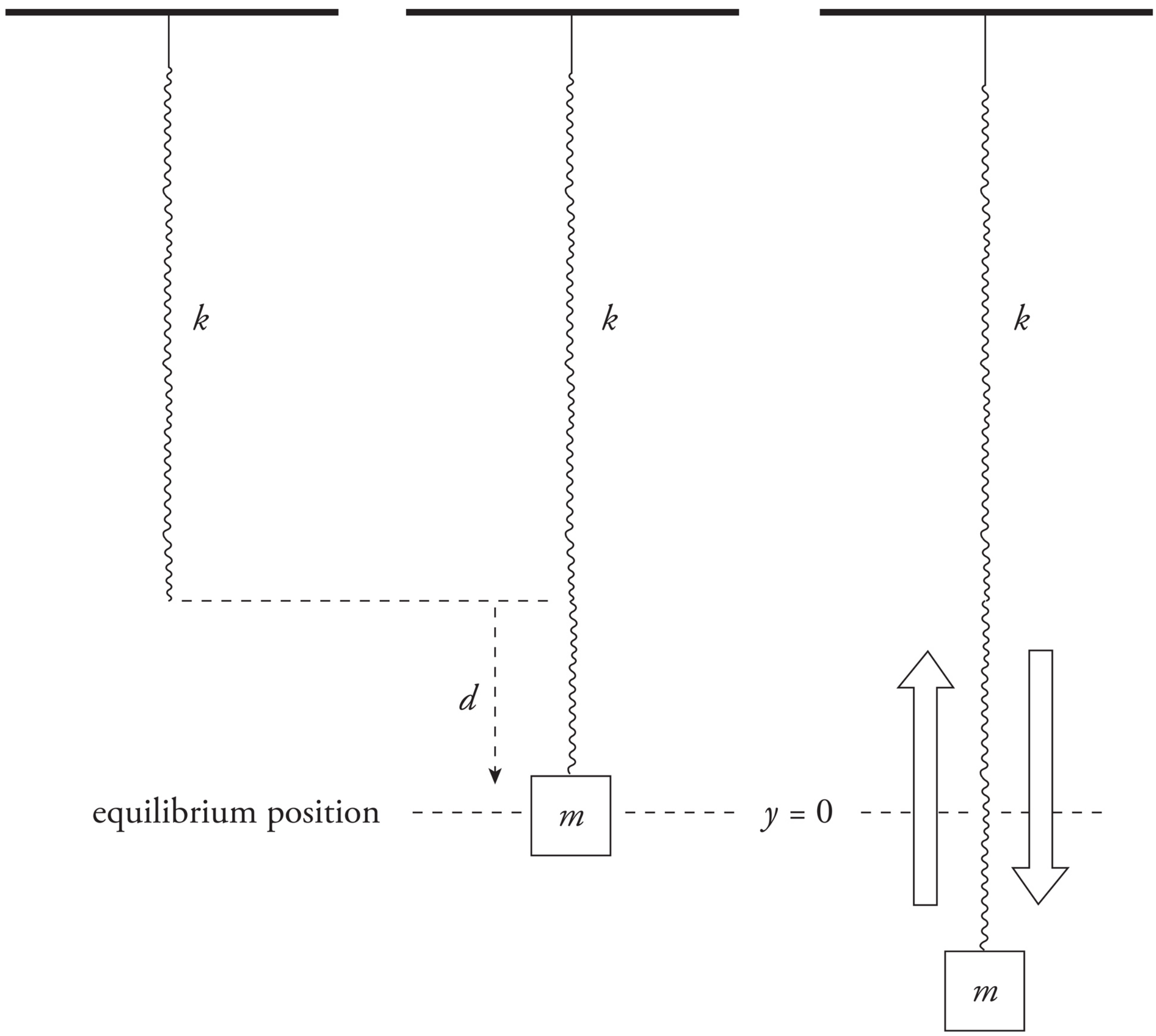
Using a force diagram, let's map out the situation described on this page:



Our net force in this case is zero since it is neither moving up nor moving down. So our force, by the spring, must cancel out with our weight: $kx = mg$.

Consider a spring of negligible mass hanging from a stationary support. A block of mass m is attached to its end and allowed to come to rest, stretching the spring a distance d. At this point, the block is in equilibrium; the upward force of the spring is balanced by the downward force of gravity. Therefore,

$$kd = mg \quad \Rightarrow \quad d = \frac{mg}{k}$$



Once you have solved for the distance d, simply set this position as your new equilibrium position $x = 0$. At this point, you can treat the vertical spring exactly as you would a horizontal spring. Just be sure to measure distances relative to the new equilibrium when solving for any values (spring force, potential energy, and so on) that have x in their formulas.

New Equilibrium
Normally our equilibrium for the spring (without a block) would be at the position y. When we attach the block, our new equilibrium point sits at $(d + y)$. The only time you need to worry about this is if a question asks about the total length of the spring at a given moment. Use your horizontal spring equations to solve the question normally, and then simply add d.

Example 11 A block of mass $m = 1.5$ kg is attached to the end of a vertical spring of force constant $k = 300$ N/m. After the block comes to rest, it is pulled down a distance of 2.0 cm and released.

(a) What is the frequency of the resulting oscillations?

(b) What are the minimum and maximum amounts of stretch of the spring during the oscillations of the block?

Solution.

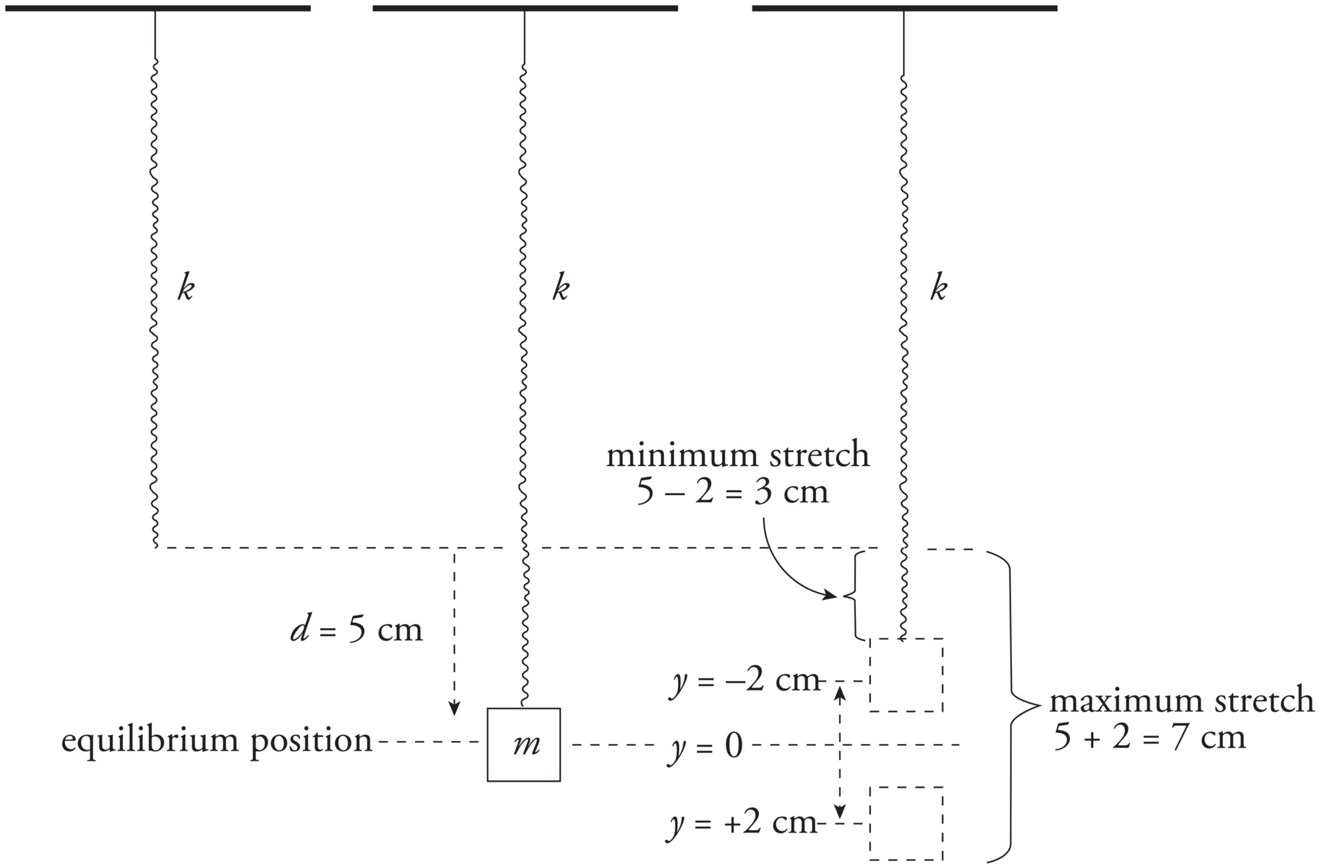
(a) The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{300 \text{ N/m}}{1.5 \text{ kg}}} = 2.3 \text{ Hz}$$

(b) Before the block is pulled down, to begin the oscillations, it stretches the spring by a distance calculated as follows:

$$d = \frac{mg}{k} = \frac{(1.5 \text{ kg})(10 \text{ N/kg})}{300 \text{ N/m}} = 0.05 \text{ m} = 5 \text{ cm}$$

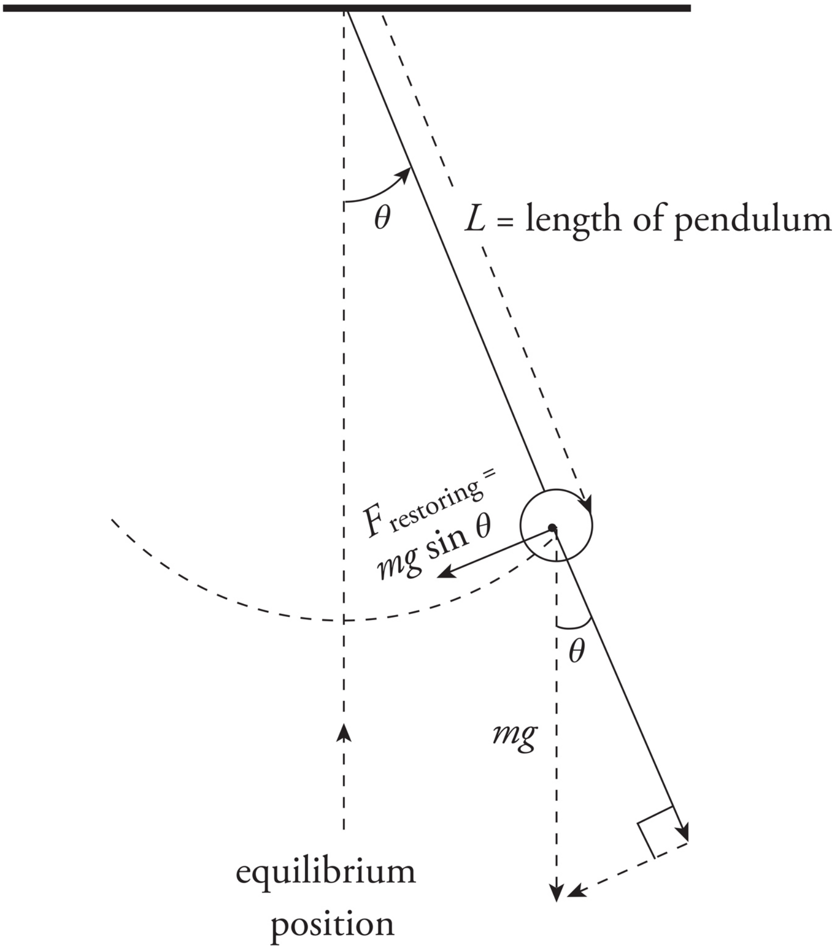
Since the amplitude of the motion is 2.0 cm, the spring is stretched a maximum of 5 cm + 2.0 cm = 7 cm when the block is at the lowest position in its cycle, and a minimum of 5 cm – 2.0 cm = 3 cm when the block is at its highest position.



PENDULUMS

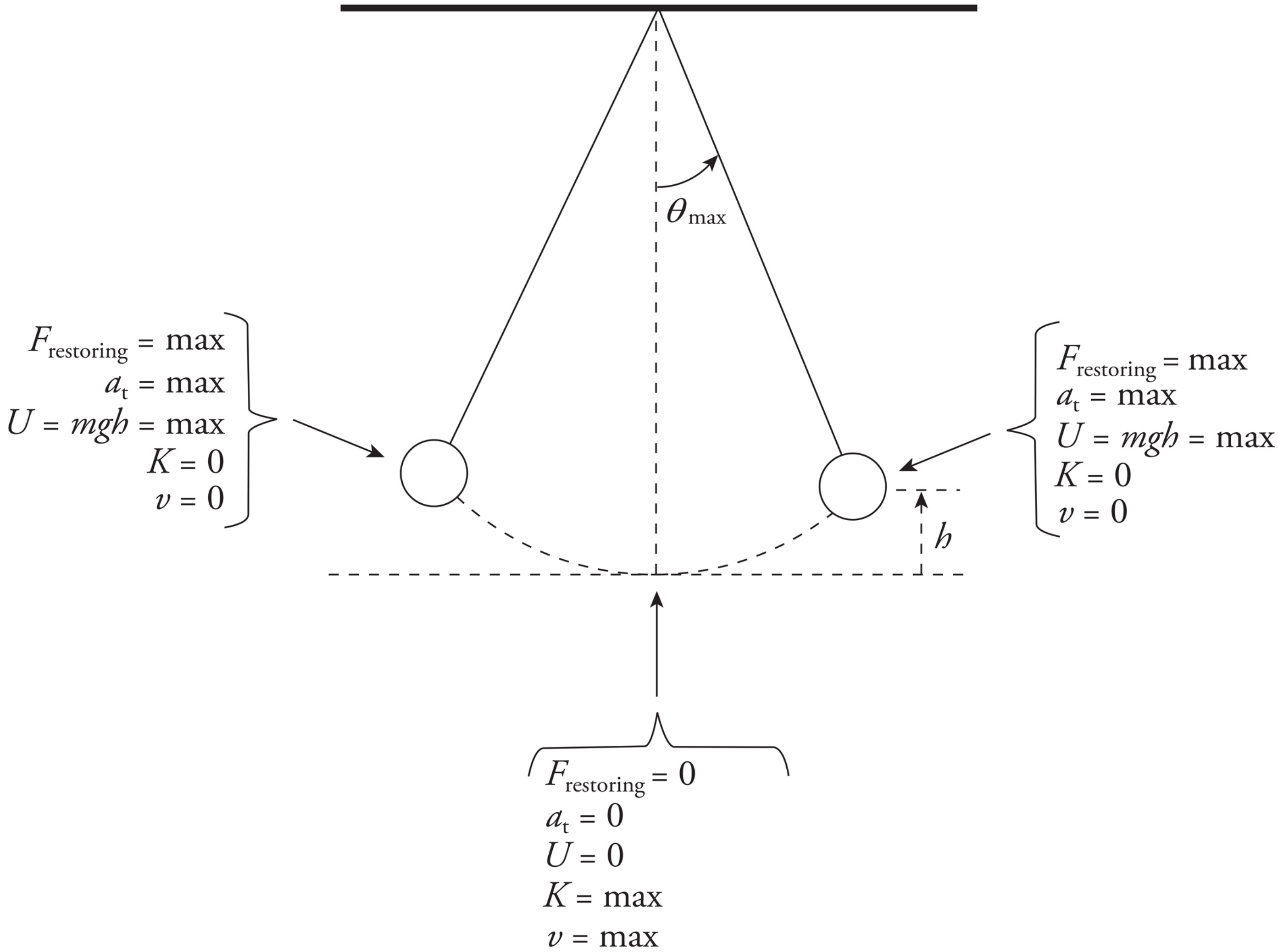
A **simple pendulum** consists of a weight of mass m attached to a string or a massless rod that swings, without friction, about the vertical equilibrium position. The restoring force is provided by gravity and, as the figure on the next page shows, the magnitude of the restoring force when the bob is θ to an angle to the vertical is given by the equation:

$$F_{\text{restoring}} = mg \sin \theta$$



Although the displacement of the pendulum is measured by the angle that it makes with the vertical, rather than by its linear distance from the equilibrium position (as was the case for the spring-block oscillator), the simple pendulum shares many of the important features of the spring-block oscillator. For example,

- Displacement is zero at the equilibrium position.
- At the endpoints of the oscillation region (where $\theta = \pm\theta_{max}$), the restoring force and the tangential acceleration (a_t) have their greatest magnitudes, the speed of the pendulum is zero, and the potential energy is maximized.
- As the pendulum passes through the equilibrium position, its kinetic energy and speed are maximized.



Despite these similarities, there is one important difference. Simple harmonic motion results from a restoring force that has a strength that is proportional to the displacement. The magnitude of the restoring force on a pendulum is $mg \sin \theta$, which is *not* proportional to the displacement (θL, the arc length, with the angle measured in radians). Strictly speaking, the motion of a simple pendulum is not really simple harmonic. However, if θ is small, then $\sin \theta \approx \theta$ (measured in radians) so, in this case, the magnitude of the restoring force is approximately $mg\theta$, which *is*proportional to θ. So if θ_{max} is small, the motion can be treated as simple harmonic.

> More so than other topics, pendulum problems are often easier to solve if θ is measured in radians than if it's measured in degrees. If you're not comfortable working in radians, convert to degrees using the conversion 180 degrees = π radians.

If the restoring force is given by $mg\theta$, rather than $mg \sin \theta$, then the frequency and period of the oscillations depend only on the length of the pendulum and the value of the gravitational acceleration, according to the following equations:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \quad \text{and} \quad T = 2\pi\sqrt{\frac{L}{g}}$$

Note that neither frequency nor the period depends on the amplitude (the maximum angular displacement, θ_{max}); this is a characteristic feature of simple harmonic motion. Also notice that neither the frequency nor the period depends on the mass of the weight.

The same mnemonic we used for springs can be applied to pendulums. The equation for frequency goes in alphabetical order (clockwise), and the equation for period goes in reverse alphabetical order (clockwise).

Example 12 A simple pendulum has a period of 1 s on Earth. What would its period be on the Moon (where g is one-sixth of its value here)?

Solution. The equation $T = 2\pi\sqrt{L/g}$ shows that T is inversely proportional to $\sqrt{g}$, so if g decreases by a factor of 6, then T increases by factor of $\sqrt{6}$. That is,

$$T_{\text{on Moon}} = \sqrt{6} \times T_{\text{on Earth}} = (1 \text{ s})\sqrt{6} = 2.4 \text{ s}$$

Summary

- $T = \frac{time}{\# \, cycles}$

  $f = \frac{\# \, cycles}{time}$

  $T = \frac{1}{f}$

- Hooke's Law holds for most springs. Formulas to keep in mind are the following:

  $F_s = -kx$

  $T = 2\pi\sqrt{\frac{m}{k}}$

  $U_s = \frac{1}{2}kx^2$

- For small angle of a pendulum swing:

  $T = 2\pi\sqrt{\frac{L}{g}}$