Linear Momentum

"Nothing will happen until something moves."

—Albert Einstein

Previously we discovered the nature of objects, the reason why objects move the way they do, and the energy required to make objects move. Now we will predict the nature of objects when they interact with other objects and the resulting outcome.

INTRODUCTION TO MOMENTUM

A pool stick hitting the cue ball, a car collision, the Death Star exploding—physics is about the interaction of objects. A collision is a complex interaction between two objects. But what happens after two objects interact with each other? If we use Newton's Second Law, it will prove to be quite a tedious challenge. The forces occurring during a car collision are unimaginably complex. And not all collisions end with the same results: sometimes they move away from each other, sometimes they move in the same direction, and sometimes the objects stick together. If we were to measure their speeds before the collision and their speeds after the collision, we would also get different results. Using our observations and a general understanding of Newton's Second Law, we could predict that the smaller car in a collision (head on and both cars traveling with the same speed) would end up having the greater speed in the end. How can such a complex interaction give rise to a simple outcome?

WHAT IS MOMENTUM?

When Newton first expressed his Second Law, he didn't write $\mathbf{F}_{net} = m\mathbf{a}$. Instead, he expressed the law in the words, *The alteration of motion is…proportional to the…force impressed*…. By "motion," he meant the product of mass and velocity, a vector quantity known as **linear momentum** and denoted by **p:**

$$p = mv$$

Momentum Units

$p = mv$ so the unit for momentum is a kg · m/s. There is no special term for this unit.

Momentum is a Vector

Remember that a vector has magnitude and direction. In collision problems, be aware of orientation and assign negative values to negative velocities and positive values for positive velocities.

So Newton's original formulation of the second law read $\Delta\mathbf{p} \propto \mathbf{F}$, or, equivalently, $\mathbf{F} \propto \Delta\mathbf{p}$. But a large force that acts for a short period of time can produce the same change in linear momentum as a small force acting for a greater period of time. Knowing this, we can turn the proportion above into an equation, if we take the average force that acts over the time interval Δt:

$$\mathrm{F} = \frac{\Delta \mathrm{p}}{\Delta t}$$

This equation becomes $\mathbf{F} = ma$, since $\Delta\mathbf{p}/\Delta t = \Delta(m\mathbf{v})/\Delta t = m\,(\Delta\mathbf{v}/\Delta t) = ma$.

Example 1 A golfer strikes a golf ball of mass 0.05 kg, and the time of impact between the golf club and the ball is 1 ms. If the ball acquires a velocity of magnitude 70 m/s, calculate the average force exerted on the ball.

> **Reminder!**
> Remember that momentum and kinetic energy are not the same thing.

Solution. Using Newton's Second Law, we find

$$\overline{F} = \frac{\Delta p}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = m\frac{v-0}{\Delta t} = (0.05\text{ kg})\frac{70\text{ m/s}}{10^{-3}\text{ s}} = 3{,}500\text{ N}$$

IMPULSE

The product of force and the time during which it acts is known as **impulse;** it's a vector quantity that's denoted by **J:**

$$\mathbf{J} = \mathbf{F}\Delta t$$

> **Newton's First Law**
> Objects naturally resist changes in their motion. In order for us to change an object's speed or direction, we must induce some kind of force over a period of time.

Impulse is equal to change in linear momentum. In terms of impulse, Newton's Second Law can be written in yet another form:

$$\mathbf{J} = \Delta\mathbf{p}$$

Sometimes this is referred to as the **Impulse–Momentum Theorem,** but it's just another way of writing Newton's Second Law. The impulse delivered to an object may be found by taking the area under a force-versus-time (also referred to as an F-versus-t) graph.

The Impulse–Momentum Theorem basically states that an impulse that is delivered on an object changes its momentum. The momentum "after" the collision is equal to the momentum "before" the collision added in with the impulse required to get your final outcome. In equation form, this states

$$p_{\text{final}} = p_{\text{initial}} + J$$

Impulse eliminates the need to use $F = ma$ and simplifies the intricate forces delivered into mass, initial, and final velocities.

> **Example 2** A football team's kicker punts the ball (mass = 0.4 kg) and gives it a launch speed of 30 m/s. Find the impulse delivered to the football by the kicker's foot and the average force exerted by the kicker on the ball, given that the impact time is 8 ms.

Solution. As we know, impulse is equal to change in linear momentum, so

$$J = \Delta p = p_f - p_i = p_f = mv = (0.4\text{ kg})(30\text{ m/s}) = 12\text{ kg·m/s}$$

Using the equation $\overline{F} = J/\Delta t$, we find that the average force exerted by the kicker is

$$\overline{F} = J/\Delta t = (12\text{ kg·m/s})/(8 \times 10^{-3}\text{ s}) = 1{,}500\text{ N}$$

Example 3 An 80 kg stuntman jumps out of a window that's 45 m above the ground.

(a) How fast is he falling when he reaches ground level?

(b) He lands on a large, air-filled target, coming to rest in 1.5 s. What average force does he feel while coming to rest?

(c) What if he had instead landed on the ground (impact time = 10 ms)?

Solution.

(a) His gravitational potential energy turns into kinetic energy: $mgh = \frac{1}{2}mv^2$, so

$$v = \sqrt{2gh} = \sqrt{2(10)(45)} = 30\text{ m/s}$$

(You could also have answered this question using Big Five #5.)

(b) Using $\mathbf{F} = \Delta\mathbf{p}/\Delta t$, we find that

$$\mathrm{F} = \frac{\Delta \mathrm{p}}{\Delta t} = \frac{\mathrm{p}_f - \mathrm{p}_i}{\Delta t} = \frac{0 - m\mathrm{v}_i}{\Delta t} = \frac{-(80\text{ kg})(30\text{ m/s})}{1.5\text{ s}} = -1{,}600\text{ N} \quad \Rightarrow \quad F = 1{,}600\text{ N}$$

(c) In this case,

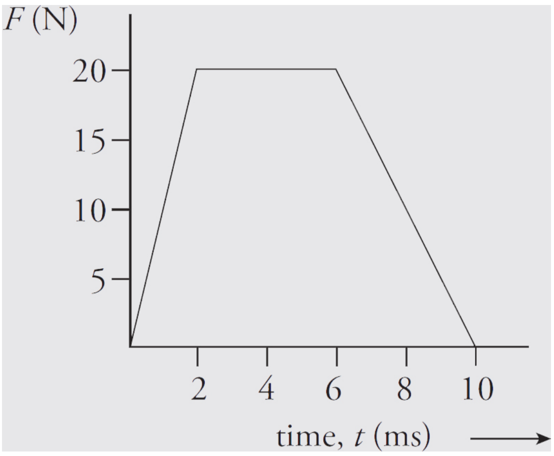
$$\mathrm{F} = \frac{\Delta \mathrm{p}}{\Delta t} = \frac{\mathrm{p}_f - \mathrm{p}_i}{\Delta t} = \frac{0 - m\mathrm{v}_i}{\Delta t} = \frac{-(80\text{ kg})(30\text{ m/s})}{10\times 10^{-3}\text{ s}} = -240{,}000\text{ N} \quad \Rightarrow \quad F = 240{,}000\text{ N}$$

The negative signs in the vector answers to (b) and (c) simply tell you that the forces are acting in the opposite direction of motion and will cause the object to slow down. This force is equivalent to about 27 tons (!), more than enough to break bones and cause fatal brain damage. Notice how crucial impact time is: increasing the slowing-down time reduces the acceleration and the force, ideally enough to prevent injury. This is the purpose of air bags in cars, for instance.

Go Online!
Remember what we mentioned earlier in the book: the College Board announced that they will be re-organizing their AP Physics 1 course content in May 2019. Go online to the College Board's website and/or your Princeton Review online Student Tools for late-breaking updates as they are shared.

Curved Graphs?
Calculus will be needed to find the true area under curved graphs. But the AP Physics 1 Exam does not require you to know calculus. You can approximate areas under curved graphs by using basic shapes you have studied before in geometry.

Example 4 A small block of mass m = 0.07 kg, initially at rest, is struck by an impulsive force F of duration 10 ms whose strength varies with time according to the following graph:



What is the resulting speed of the block?

Solution. The impulse delivered to the block is equal to the area under the F-versus-t graph. The region is a trapezoid, so its area, $\frac{1}{2}$ (base$_1$ + base$_2$) × height, can be calculated as follows:

$$J = \frac{1}{2}[(10\text{ ms} - 0) + (6\text{ ms} - 2\text{ ms})] \times (20\text{ N}) = 0.14\text{ N} \cdot \text{s}$$

Now, by the Impulse–Momentum Theorem,

$$J = \Delta p = p_f - p_i = mv \quad \Rightarrow \quad v_f = \frac{J}{m} = \frac{0.14\text{ N} \cdot \text{s}}{0.07\text{ kg}} = 2\text{ m/s}$$

I Found the Area, but What Does It Mean?
Whenever you find the area under a curve, you're essentially multiplying the quantities on the two axes. In this example, the area represents the product of force and time. That's how we know we're dealing with impulse.

CONSERVATION OF LINEAR MOMENTUM

Newton's Third Law states that when one object exerts a force on a second object, the second object exerts an equal but opposite force on the first. Newton's Third Law combines with Newton's Second Law when two objects interact with each other.

In the previous section, we redefined Newton's Second Law as Impulse–Momentum Theorem, $J = \Delta p$. If we combine the laws and interpret them in terms of momentum, it states that two interacting objects experience equal but opposite momentum changes (assuming we have an isolated system, meaning no external forces).

The total linear momentum of an isolated system remains constant.

The momentum "before" equals the momentum "after." This is the Law of Conservation of Momentum, which states

$$\text{total } \mathbf{p}_{\text{initial}} = \text{total } \mathbf{p}_{\text{final}}$$

No Exceptions!
Just like conservation of energy, the Law of Conservation of Momentum is a universal law. There are no special circumstances that allow for violations. While it won't always be helpful to consider that momentum is being conserved, it is nevertheless true in every situation.

Example 5 An astronaut is floating in space near her shuttle when she realizes that the cord that's supposed to attach her to the ship has become disconnected. Her total mass (body + suit + equipment) is 91 kg. She reaches into her pocket, finds a 1 kg metal tool, and throws it out into space with a velocity of 9 m/s directly away from the ship. If the ship is 10 m away, how long will it take her to reach it?

Solution. Here, the astronaut + tool are the system. Because of Conservation of Linear Momentum,

$$
\begin{aligned}
m_{\text{astronaut}}\mathbf{v}_{\text{astronaut}} + m_{\text{tool}}\mathbf{v}_{\text{tool}} &= 0 \\
m_{\text{astronaut}}\mathbf{v}_{\text{astronaut}} &= -m_{\text{tool}}\mathbf{v}_{\text{tool}} \\
\mathbf{v}_{\text{astronaut}} &= -\frac{m_{\text{tool}}}{m_{\text{astronaut}}}\mathbf{v}_{\text{tool}} \\
&= -\frac{1\ \text{kg}}{90\ \text{kg}}(-9\ \text{m/s}) = +0.1\,\text{m/s}
\end{aligned}
$$

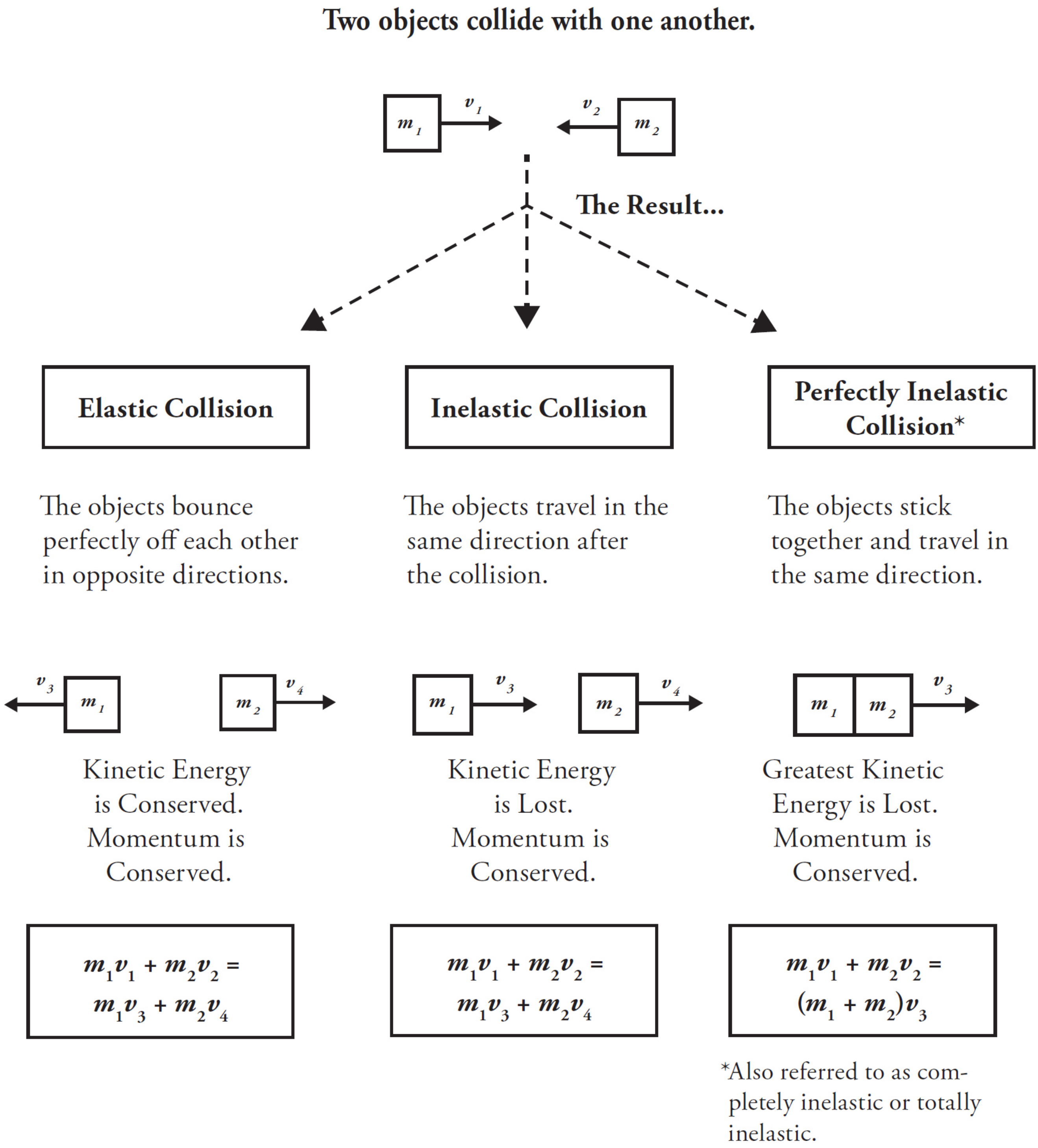
Using *distance* = *rate* × *time*, we find

$$
t = \frac{d}{v} = \frac{10\ \text{m}}{0.1\ \text{m/s}} = 100\ \text{s}
$$

COLLISIONS

Conservation of Linear Momentum is routinely used to analyze **collisions.** The objects whose collision we will analyze form the *system*, and although the objects exert forces on each other during the impact, these forces are only *internal* (they occur within the system), and the system's total linear momentum is conserved.

Let's break down the collision types:



Example 6 Two balls roll toward each other. The red ball has a mass of 0.5 kg and a speed of 4 m/s just before impact. The green ball has a mass of 0.2 kg and a speed of 2 m/s. After the head-on collision, the red ball continues forward with a speed of 2 m/s. Find the speed of the green ball after the collision. Was the collision elastic?

Solution. First, remember that momentum is a vector quantity, so the direction of the velocity is crucial. Since the balls roll toward each other, one ball has a positive velocity while the other has a negative velocity. Let's call the red ball's velocity before the collision positive; then $v_{red} = +4$ m/s, and $v_{green} = -2$ m/s.

> **Remember Vectors!**
> Momentum is a vector. If you set an object traveling to the right as the positive momentum, you must give an object going to the left a negative momentum.

Using a prime to denote *after the collision*, Conservation of Linear Momentum gives us the following:

$$\begin{aligned} \text{total } \mathbf{p}_{\text{before}} &= \text{total } \mathbf{p}_{\text{after}} \\ m_{\text{red}}\mathbf{v}_{\text{red}} + m_{\text{green}}\mathbf{v}_{\text{green}} &= m_{\text{red}}\mathbf{v}'_{\text{red}} + m_{\text{green}}\mathbf{v}'_{\text{green}} \\ (0.5)(+4)+(0.2)(-2) &= (0.5)(+2)+(0.2)\mathbf{v}'_{\text{green}} \\ \mathbf{v}'_{\text{green}} &= +3.0 \text{ m/s} \end{aligned}$$

Notice that the green ball's velocity was reversed as a result of the collision; this typically happens when a lighter object collides with a heavier object. To see whether the collision was elastic, we need to compare the total kinetic energies before and after the collision.

Initially,

$$K_t = K_1 + K_2$$

$$= \frac{1}{2}m_1 v_{i1}^2 + \frac{1}{2}m_2 v_{i2}^2$$

$$= \frac{1}{2}(0.5)(+4)^2 + \frac{1}{2}(0.2)(-2)^2$$

$$= 4.4 \text{ J}$$

There are 4.4 joules at the beginning. At the end,

$$K_t = K_1 + K_2$$

$$= \frac{1}{2}m_1 v_{f1}^2 + \frac{1}{2}m_2 v_{f2}^2$$

$$= \frac{1}{2}(0.5)(+2)^2 + \frac{1}{2}(0.2)(3)^2$$

$$= 1.9 \text{ J}$$

> **Elastic Versus Inelastic**
> All collisions conserve momentum. In order to determine whether an unknown collision was elastic or inelastic, we must check if kinetic energy was lost. If kinetic energy was lost, then the collision was inelastic.

So, there is less kinetic energy at the end compared to the beginning. Kinetic energy was lost (so the collision was inelastic). Most of the lost energy was transferred as heat; the two objects are both slightly warmer as a result of the collision.

Example 7 Two balls roll toward each other. The red ball has a mass of 0.5 kg and a speed of 4 m/s just before impact. The green ball has a mass of 0.3 kg and a speed of 2 m/s. If the collision is completely inelastic, determine the velocity of the composite object after the collision.

Solution. If the collision is completely inelastic, then, by definition, the masses stick together after impact, moving with a velocity, v′. Applying Conservation of Linear Momentum, we find

$$\begin{aligned} \text{total } \mathbf{p}_{\text{before}} &= \text{total } \mathbf{p}_{\text{after}} \\ m_{\text{red}}\mathbf{v}_{\text{red}} + m_{\text{green}}\mathbf{v}_{\text{green}} &= (m_{\text{red}} + m_{\text{green}})\mathbf{v}' \\ (0.5)(4) + (0.3)(-2) &= (0.5 + 0.3)\mathbf{v}' \\ \mathbf{v}' &= +1.8 \text{ m/s} \end{aligned}$$

Example 8 A 500 kg car travels 20 m/s due north. It hits a 500 kg car traveling due west at 30 m/s. The cars lock bumpers and stick together. What is the velocity the instant after impact?

Solution. This problem illustrates the vector nature of numbers. First, look only at the x (east-west) direction. There is only one car moving west and its momentum is given by

$$\mathbf{p} = m\mathbf{v} \Rightarrow (500 \text{ kg})(30 \text{ m/s}) = 15{,}000 \text{ kg} \cdot \text{m/s west}$$

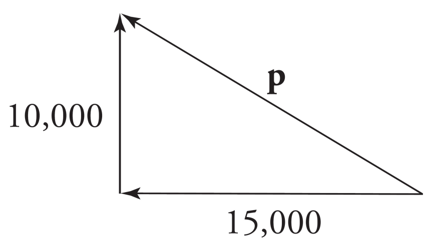
Next, look only at the y (north-south) direction. There is only one car moving north and its momentum is given by

$$\mathbf{p} = m\mathbf{v} \Rightarrow (500 \text{ kg})(20 \text{ m/s}) = 10{,}000 \text{ kg} \cdot \text{m/s north}$$

The total final momentum is the resultant of these two vectors. Use the Pythagorean Theorem:

$$(10{,}000)^2 + (15{,}000)^2 = \mathbf{p}_f^2$$

$$\mathbf{p} = 18{,}027 \text{ kg} \cdot \text{m/s}$$



To find the total velocity you need to solve for v using

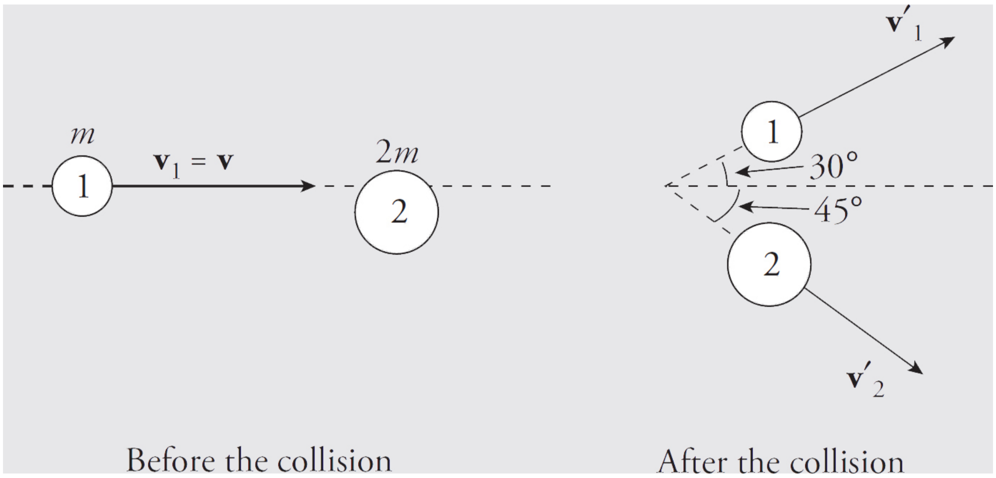
$$\mathbf{p}_f = mv \Rightarrow 18{,}027 \text{ kg} \cdot \text{m/s} = (500 \text{ kg} + 500 \text{ kg})(v)$$

$$v = 18 \text{ m/s}$$

However, we are not done yet because velocity has both a magnitude (which we now know is 18 m/s) and a direction. The direction can be expressed using tan θ = 10,000/15,000 so θ =

$\tan^{-1}(10{,}000/15{,}000)$, or 33.7 degrees north of west. Again, most of the AP Physics 1 Exam is in degrees, not radians.

Example 9 An object of mass m moves with velocity $\mathbf{v}$ toward a stationary object of mass $2m$. After impact, the objects move off in the directions shown in the following diagram:



(a) Determine the magnitudes of the velocities after the collision (in terms of v).

(b) Is the collision elastic? Explain your answer.

Solution.

(a) Conservation of Linear Momentum is a principle that establishes the equality of two vectors: $\mathbf{p}_{\text{total}}$ before the collision and $\mathbf{p}_{\text{total}}$ after the collision. Writing this single vector equation as two equations, one for the x-component and one for the y, we have

$$x\text{-component:}\quad mv = mv_1'\cos 30^\circ + 2mv_2'\cos 45^\circ \quad (1)$$

$$y\text{-component:}\quad 0 = mv_1'\sin 30^\circ - 2mv_2'\sin 45^\circ \quad (2)$$

Adding these equations eliminates v_2', because $\cos 45^\circ = \sin 45^\circ$.

$$mv = mv'_1(\cos 30^\circ + \sin 30^\circ)$$

and lets us determine v_1':

$$v_1' = \frac{v}{\cos 30^\circ + \sin 30^\circ} = \frac{2v}{1+\sqrt{3}}$$

Substituting this result into Equation (2) gives us

$$0 = m\frac{2v}{1+\sqrt{3}}\sin 30^\circ - 2mv_2'\sin 45^\circ$$

$$2mv_2'\sin 45^\circ = m\frac{2v}{1+\sqrt{3}}\sin 30^\circ$$

$$v_2' = \frac{\frac{2v}{1+\sqrt{3}}\sin 30^\circ}{2\sin 45^\circ} = \frac{v}{\sqrt{2}(1+\sqrt{3})}$$

> The math will rarely (if ever) be this intense on the actual exam. Don't stress too much if this problem took you a few minutes to work through. We'd rather you be over-prepared!

(b) The collision is elastic only if kinetic energy is conserved. The total kinetic energy after the collision, K', is calculated as follows:

$$\begin{aligned}
K' &= \frac{1}{2}\cdot mv_1'^2 + \frac{1}{2}\cdot 2mv_2'^2 \\
&= \frac{1}{2}m\left(\frac{2v}{1+\sqrt{3}}\right)^2 + m\left(\frac{v}{\sqrt{2}(1+\sqrt{3})}\right)^2 \\
&= mv^2\left[\frac{2}{(1+\sqrt{3})^2} + \frac{1}{2(1+\sqrt{3})^2}\right] \\
&= \frac{5}{2(1+\sqrt{3})^2}mv^2
\end{aligned}$$

However, the kinetic energy before the collision is just $K = \frac{1}{2}mv^2$, so the fact that

$$\frac{5}{2(1+\sqrt{3})^2} < \frac{1}{2}$$

tells us that K' is less than K, so some kinetic energy is lost; the collision is inelastic.

Summary

- Momentum is a vector quantity given by $\mathbf{p} = m\mathbf{v}$**.** If you push on an object for some amount of time we call that an impulse **(J).** Impulse causes a change in momentum. Impulse is also a vector quantity and these ideas are summed up in the equations $\mathbf{J} = \mathbf{F}\Delta t$ or $\mathbf{J} = \Delta p$.

- Momentum is a conserved quantity in a closed system (that is, a system with no external forces). That means

$$\text{total } \mathbf{p}_\text{i} = \text{total } \mathbf{p}_\text{f}$$

- Overall strategy for conservation of momentum problems:
    - I. Create a coordinate system.
    - II. Break down each object's momentum into x- and y-components. That is $p_x = p\cos\theta$ and $p_y = p\sin\theta$ for any given object.
    - III. $\sum p_{xi} = \sum p_{xf}$ and $\sum p_{yi} = \sum p_{yf}$
    - IV. Sometimes you end up rebuilding vectors in the end. Remember total $p = \sqrt{p_x{}^2 + p_y{}^2}$ and the angle is given by $\theta = \tan^{-1}\left(\dfrac{p_y}{p_x}\right)$.